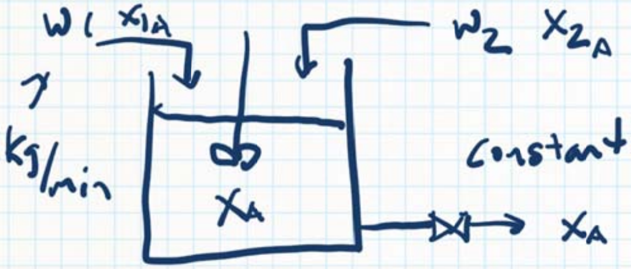
Lecture 3 :

8.28.18

substance A



constant volume

$$\underbrace{\frac{dV}{dt}}_{\text{Acc.}} = \frac{1}{\rho}(\underbrace{w_1 + w_2}_{\text{mass in}} - \underbrace{w}_{\text{mass out}})$$

$$\frac{dx}{dt} = \frac{w_1}{\rho V}\left[x_1 - x\right] + \frac{w_2}{\rho V}\left[x_2 - x\right]$$

$$\underbrace{\tau}_{\text{time constant}} \frac{dx}{dt} + x = C$$

What is this?

First order linear ODE

$\underline{x(0)} = 0.5$

How do we solve these?

* Analytically (Math courses) → separation of variables
  Integrating Factor

* Numerically (310) → Matlab, Python
  Excel

Separation of Variables Solution

$$\tau \frac{dx}{dt} = c - x$$

$$\int_{x(0)}^{x} \frac{1}{c-x}\,dx = \int_0^t \frac{1}{\tau}\,dt \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b)$$

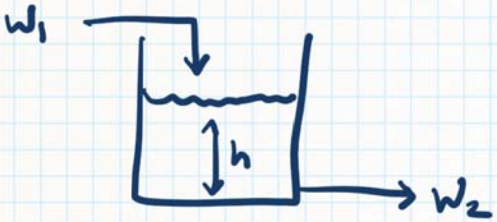
$$-\ln(c-x) + \ln(c-x(0)) = \frac{1}{\tau}(t-0)$$

$$\ln\left(\frac{c-x}{c-x(0)}\right) = -\frac{t}{\tau}$$

$$c - x = (c - x(0))\exp(-t/\tau)$$

$$x = c - (c - x(0))\exp(-t/\tau)$$

Dynamic Model # 2 = Liquid Storage



$V = \underset{\substack{\uparrow \\ \text{constant}}}{A}h$

$$\text{Mass Accum} = \text{Mass in} - \text{Mass out}$$

$$\frac{d\rho V}{dt} = \rho q_1 - \rho q_2$$

ρ = constant
$V = Ah$

q_1: $\frac{m^3}{s}$ (volumetric flowrate)

(a) No flow in, only flow out

$$\frac{dh}{dt} = -\frac{q}{A} \rightarrow h(t) = h(0) - \frac{q}{A}t$$

(b) Valve inserted on outlet; flow out is proportional to height

$$q_2 = \frac{1}{R_V} h$$

$$A\frac{dh}{dt} = q_1 - \frac{h}{R_V}$$

$$\underbrace{AR_V}_{\tau} \frac{dh}{dt} + h = \underbrace{q_1 R_V}_{c}$$

linear ODE

integrating process
NOT self-regulation

(c) Flow rate out is dependent on Pressure drop

Bernoulli's Eqn.

pressure at bottom of tank

$$q_2 = C_v^* \sqrt{\frac{P - P_a}{\rho}} \qquad P = P_a + \rho g h$$

$$q_2 = C_v^* \sqrt{gh}$$

$$= c\sqrt{h}$$

$$A \frac{dh}{dt} = q_1 - c\sqrt{h}$$

← non-linear form (harder to solve analytically.)

→ CSTR → homework

(1) Draw picture

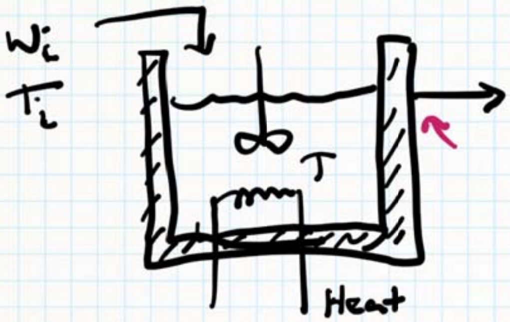
(2) General balance

(3) Insert terms, simplify

(4) Solve → analytical or numerical

Energy Balance

CV= Temperature



Assumptions

(1) Perfect mixing

(2) Constant volume

(3) ρ and C_p constants (NOT Temp dependent)

(4) Negligible heat loss

(5) Negligible shaft work

(6) Enthalpy at moderate pressure
neglect pressure effects on Enthalpy (H)

$$\underbrace{\frac{dU}{dt}}_{} = \underbrace{w_i \hat{H}_i}_{\text{energy flowing in}} - \underbrace{w\hat{H}}_{\text{energy flows out}} + \dot{Q}$$

Heat aka generation term

Appendix B

$\hat{U} = \hat{H}$

$$\frac{d(\rho V \hat{U})}{dt} = \rho \left[\hat{U}\frac{dV}{dt} + V\frac{d\hat{U}}{dt} \right]$$

$\frac{dV}{dt} = 0$

$$\hat{U} = \hat{C}_V\, dT = \hat{C}_p\, dT$$

just for liquids

constant volume energy balance

$$\rho V \hat{C}_v \frac{dT}{dt} = w_1 \hat{H}_1 - w_1 \hat{H} + \dot{Q}$$

$$\rho V \hat{C}_p \frac{dT}{dt} = w_1 (\hat{H}_i - \hat{H}) + \dot{Q}$$

def. Enthalpy $\hat{H} - \hat{H}_{ref} = C(T - T_{ref})$

heat capacity

Temp variable

$$\boxed{\rho V \hat{C}_p \frac{dT}{dt} = w_i \hat{C}_p (T_i - T) + \dot{Q}}$$

For constant volume

rearrange into this form

$$\underbrace{\frac{\rho V}{w_i}}_{\tau} \frac{dT}{dt} + T = T_i + \frac{1}{w_i \hat{C}_p} \dot{Q}$$

Control Variable

disturbance variable

manipulated variables

$$\frac{\frac{kg}{m^3} \cdot m^3}{\frac{kg}{s}} : s$$

Volume varying w/ time

$$\boxed{\rho \frac{dV}{dt} = w_i - w} \text{ (NOT ZERO)}$$

$$\hat{u}(w_i - w) + \rho V \hat{C}_v \frac{dT}{dt} = w_i \hat{H}_i - w\hat{H} + \dot{Q}$$ ← rate

↑

$\hat{H} = \hat{u} + \cancel{p\bar{V}}$ liquids

$$\rho V C_v \frac{dT}{dt} = w_i \hat{H}_i - \cancel{w\hat{H}} - w_i \hat{H} + \cancel{w\hat{H}} + \dot{Q}$$

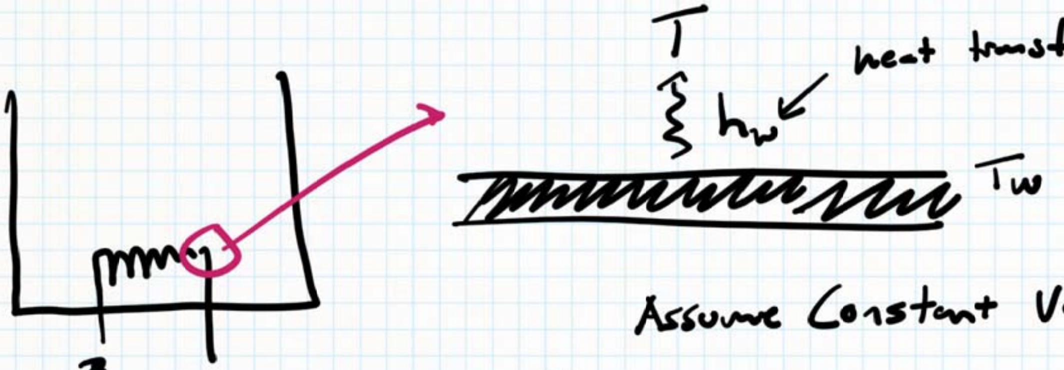
$$\rho V \hat{C}_p \frac{dT}{dt} = w_i \hat{H}_i - w_i \hat{H} + \dot{Q}$$

$$\boxed{\rho V C_p \frac{dT}{dt} = C_p (w_i T_i - w_i T) + \dot{Q}}$$ ← Variable Hold Volume

↑ first order ODE

Book has:

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q}{\rho C V}$$



heating element has heat capacity

$\dot{Q}$

Assume Constant Volume

① $$\rho V \hat{C}_p \frac{dT}{dt} = W\hat{C}_p T_i - W\hat{C}_p T + h_w \underset{\substack{\uparrow \\ \text{area of} \\ \text{heating element}}}{A_w} (T_w - T)$$

② Balance for Heater

$$\underset{\substack{\nearrow \\ \text{mass} \\ \text{of heater}}}{m_h} \underset{\substack{\uparrow \\ \text{heat capacity} \\ \text{heater}}}{C_h} \frac{dT_w}{dt} = \dot{Q}_{heater} - h_w A_w (T_w - T)$$

$\dot{Q}_{heater} = RI^2$ (R: Resistance, I: current)

Solve ① for T_w

$$T_w = \underbrace{\frac{\rho V \hat{C}_p}{h_w A_w}}_{A} T' + \underbrace{\frac{W\hat{C}_p}{h_w A_w}}_{B} T_i = BT$$

Coupled ODEs

$$T_w = AT' + BT_i + BT \quad *$$

$$T_w' = AT'' + BT' \quad * \rightarrow \text{place it in eq } 2$$

$$M_h C_{ph}\left[AT'' + BT'\right] = RI^2 - h_w A_w \left(AT' + BT_i + BT - T\right)$$

↑ How do we solve 2nd Order ODE

↳ characteristic Equations → analytical solution

→ we will use LaPlace Transforms

→ ★ CSTR

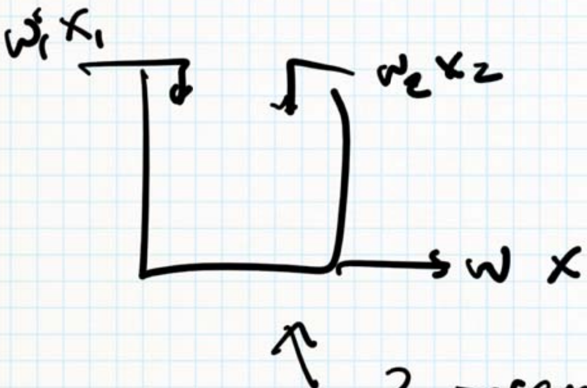
★ Multi-Stage Distillation Problem (310)

→ ★ Fed Batch Bioreactor

Degree of Freedom

→ Can a system of $\underbrace{\text{Equations}}_{}$ be solved? (model)

→ Is the solution unique?



$$\frac{\#\text{ of independent Variables}}{\#\text{ of independent model equations}}$$

$(IV - ME) = 0$ ← unique solution "exactly specified"

> 1 underspecified

< 1 overspecified

2 parameter = V, ρ

4 variables (independent) = x, x_1, w_1, w_2

1 equation = mass balance

$4 - 1 = 3$ ← Three variables must be defined to find solution

x : CV (solve)

① W_2 : MV

② x_1 ③ w_1 : DV

Lecture 4: Laplace Transforms (aka. <u>really</u> cool way to solve ODE)

Time domain

$f(t)$

ODE (<u>Linear</u>)

must be $t \geq 0$

$$f(t) \xrightarrow{\mathcal{L}[f(t)]} F(s) \qquad F(s) \xrightarrow{\mathcal{L}^{-1}[F(s)]} f(t)$$

Frequency Domain (Laplace Domain) (flea)

$F(s)$ ↑ complex variable

Algebra

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)\exp(-st)\,dt = F(s)$$

constant $f(t) = a$

$$\mathcal{L}(f(t)) = a\int_0^\infty e^{-st}\,dt$$

$$= -\frac{a}{s}e^{-st}\Big|_0^\infty$$

$$= \frac{a}{s}$$

Linear: (ramp) $\mathcal{L}(at) = a\int_0^\infty \underbrace{te^{-st}}_{\text{integrate by parts}}\,dt$

$$= F(s) = \frac{a}{s^2}$$

Properties of Laplace Transform

- Linear operator (distributive, associative)

$$\mathcal{L}(a x(t) + b y(t)) = \mathcal{L}(a(x(t))) + \mathcal{L}(b y(t))$$

$$= a \mathcal{L}(x(t)) + b \mathcal{L}(y(t))$$

$$= a X(s) + b Y(s)$$

Steps to use Laplace to solve ODE

(1) Transorm ODE to Laplace domain

(2) Apply your I.C. in the Laplace domain

(3) solve algebraically

(4) Transform back to time domain

$\boxed{\text{Ex 1}}$

$$5\frac{dy}{dt} + 4y_{(t)} = 2 \qquad \text{I.C.} \quad y(0) = 1$$

$$\mathcal{L}(a) \rightarrow \frac{a}{s}$$

$$5\left[s\,Y(s) - y(0)\right] + 4\,Y(s) = \frac{2}{s}$$

$$5s\,Y(s) - 5 + 4\,Y(s) = \frac{2}{s}$$

$$Y(s)\left[4 + 5s\right] = 5 + \frac{2}{s}$$

$$Y(s) = \frac{5 + 2/s}{(4 + 5s)} = \frac{5s + 2}{s(4 + 5s)}$$

divide by $\underline{5}$

$$Y(s) = \frac{(s + 0.4)}{s(s + 0.8)} = \frac{1}{(s + 0.8)} + \frac{0.4}{s(s + 0.8)}$$

← return to time domain

#11 on table

$b_3 = 0.4$, $b_1 = 0$, $b_2 = 0.8$

$$\underbrace{y(t) = \frac{1}{2} + \frac{1}{2}\exp(-0.8t)}_{\text{plot in Matlab}}$$

$\boxed{\text{Ex2}}$ $\dfrac{d^3y}{dt^3} + 6\dfrac{d^2y}{dt^2} + 11\dfrac{dy}{dt} + 6y = 1 \qquad y(0) = y'(0) = y''(0) = 0$

$$Y(s)\underbrace{\left(s^3 + 6s^2 + 11s + 6\right)}_{\text{characteristic eqn.}} = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^3+6s^2+11s+6)} = \frac{1}{s(s+3)(s+2)(s+1)}$$

(olive)
Heaviside Expansion

$$= \frac{\alpha_1}{s} + \frac{\alpha_2}{(s+3)} + \frac{\alpha_3}{(s+2)} + \frac{\alpha_4}{(s+1)}$$

$$\left.\frac{\cancel{(s+1)}}{s(s+3)(s+2)\cancel{(s+1)}}\right|_{s=-1} = \left.\frac{\alpha_1 \cancel{(s+1)}}{\cancel{s}} + \frac{\alpha_2 \cancel{(s+1)}}{\cancel{s+3}} + \frac{\alpha_3 \cancel{(s+1)}}{\cancel{(s+2)}} + \frac{\alpha_4 \cancel{(s+1)}}{\cancel{(s+1)}}\right|_{s=-1}$$

$$\left.\frac{1}{s(s+3)(s+2)}\right|_{s=-1} = \alpha_4 = -\frac{1}{2}$$

Partial Fraction Exp. (PFE)

$$\alpha_3 = \left.\frac{1}{s(s+3)(s+1)}\right|_{s=-2} = \frac{1}{2}$$

$$\alpha_2 = \left.\frac{1}{s(s+1)(s+2)}\right|_{s=-3} = -\frac{1}{6}$$

$$\alpha_1 = \left.\frac{1}{(s+3)(s+2)(s+1)}\right|_{s=0} = \frac{1}{6}$$

$$F(s) = \underbrace{\frac{1/6}{s} + \frac{-1/6}{(s+3)} + \frac{1/2}{(s+2)} + \frac{-1/2}{(s+1)}}_{\text{3.1 Table}}$$

$$\frac{1}{(s+b)}$$

$$f(t) = \frac{1}{6} - \frac{1}{6}e^{-3t} + \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$$

* A couple complexities w/ PFE

(1) $$Y(s) = \frac{s+1}{s(s^2+4s+4)} = \frac{s+1}{s(s+2)^2} = \frac{\alpha_1}{s} + \frac{\alpha_2}{(s+2)^2} + \frac{\alpha_3}{(s+2)}$$

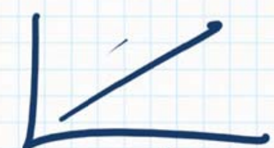
(2) (s^2+b) in denominator $\Rightarrow \dfrac{\alpha_1 s + \alpha_2}{(s^2+b)}$

Additional things we can do w/ Laplace

① S.S. solution

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \left[s\,Y(s) \right]$$

* Naturally, does not work for 'unbounded' functions



② Find initial values

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} \left[s\,Y(s) \right]$$

③ Laplace of an integral is possible (see #27 Table 3.1)

Laplace can also model time delay perturbations

$$f_d = f(t - t_0)\, S(t - t_0)$$

"unit step function"

"switch"

heaviside

occurs after t_0 : event time

$<0 \quad S = 0$

$>0 \quad S = 1$

$$F_d(s) = \mathcal{L}(f_d(t)) = \underline{e^{-st_0}}\, F(s)$$

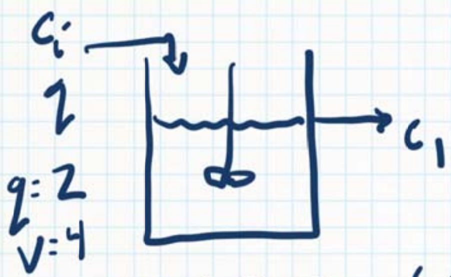
Quick Example :

$$Y(s) = \frac{1 + e^{-2s}}{(4s+1)(3s+1)} = Y_1(s) + Y_1(s)\, e^{\overbrace{-2s}^{\text{Time delay}}}$$

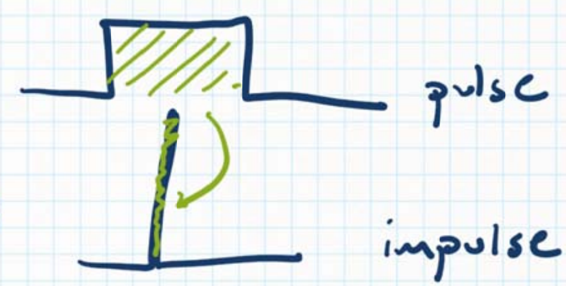
$$Y_1(s) = \frac{4}{(4s+1)} - \frac{3}{(3s+1)}$$

$$Y(t) = e^{-t/4} - e^{-t/3} + \underbrace{\left(e^{-(t-2)/4} - e^{-(t-2)/3}\right)}_{\text{time delay}} S(t-2)$$

Class 5 : Laplace Transforms ⇨ Transfer Functions



- Stirred mixer
- Change C_i
- model different "forcing functions"



Model Equation (ODE)

$$V\frac{dC_1}{dt} = q(C_i - C)$$

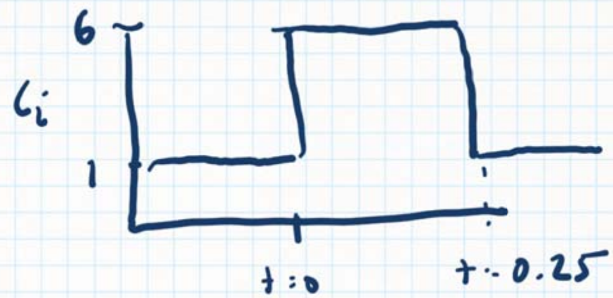
s.s. $\frac{dC_1}{dt} = 0$ $\quad C_{i_{ss}} = C_{1_{ss}} = 1$

deviation variables (perturbation variable)

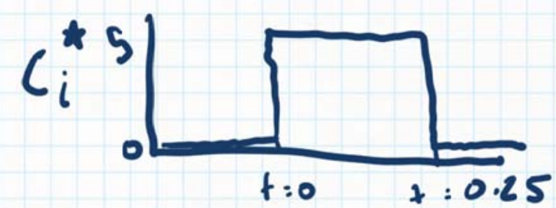
$$C_i^* = C_i - \overline{C_i}$$

true

$$C_1^* = C_1 - \overline{C_1}$$



How to write an Equation for this? →



$5S(t) - 5S(t - 1/4)$ →

$$\mathcal{L}(ODE) \quad 2sC_1(s) + C(s) = \frac{5}{s}\cancel{e}^{1} - \frac{5}{s}e^{-\frac{1}{4}s}$$

↑ time delayed solution

$$C_1(s) = \frac{5(1-e^{-s/4})}{s(2s+1)} = \frac{5}{s(2s+1)} - \frac{5e^{-s/4}}{s(2s+1)}$$

$$\frac{\alpha_1}{s} + \frac{\alpha_2}{2s+1}$$

$$C_1(s) = 5\left(\frac{1}{s} - \frac{2}{2s+1}\right) - 5e^{-s/4}\left(\frac{1}{s} - \frac{2}{2s+1}\right)$$

time is shifted by 1/4 s

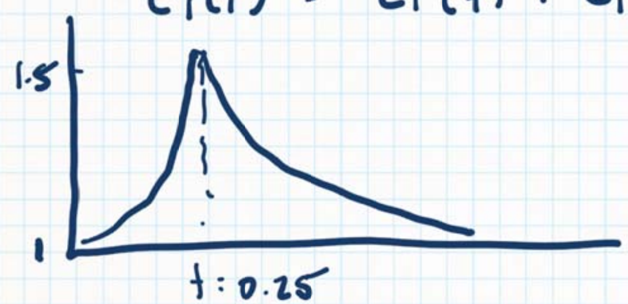
inv laplace

$$C_1^*(t) = 5(1-e^{-t/2}) - 5(1-e^{-(t-1/4)/2})\,S(t-1/4)$$

← heaviside function (switch)

$$C_1(t) = C_1^*(t) + \bar{C}_1$$

↖ 1



Same problem → impulse forcing function

δ ← dirac function

Amount of material in pulse $\frac{1}{4} 5 = \frac{5}{4}$

$$C_i = 1 + 1.25\,\delta \qquad C_i^* = 1.25\,\delta$$

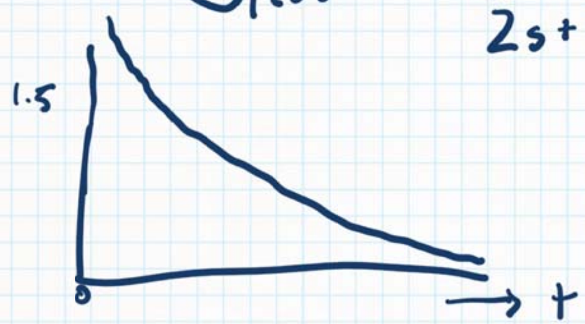
$$\underset{4}{V}\frac{dC_1^*}{dt} = \underset{2}{q}\left(C_i^* - C^*\right)$$

$\mathcal{L}$(ODE)

$$C_{1(s)}\,(2s+1) = 1.25$$

$$C_{1(s)} = \frac{1.25}{2s+1} = \frac{0.625}{s + 1/2} \xrightarrow{\text{inv Laplace}} C_1^*(t) = 0.625e^{-t/2}$$

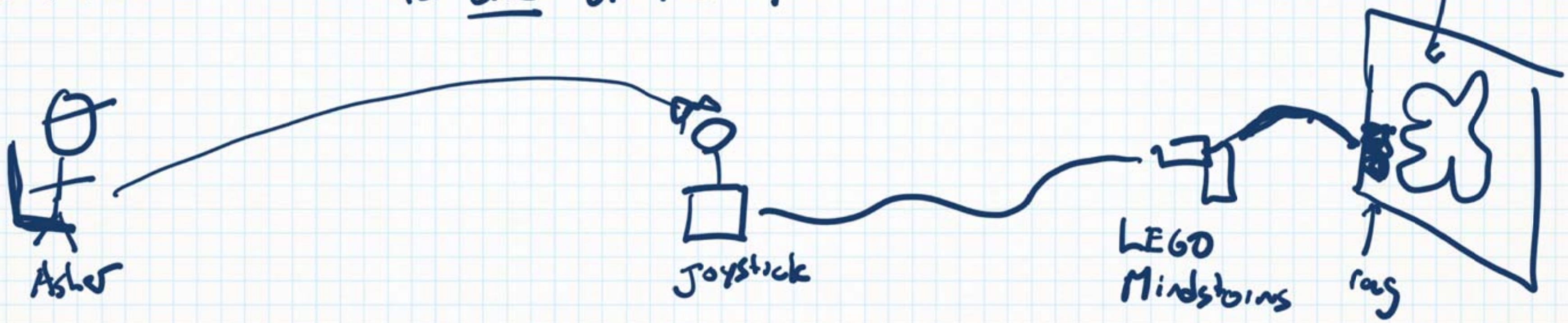
$$C_1(t) = 1 + 0.625e^{-t/2}$$



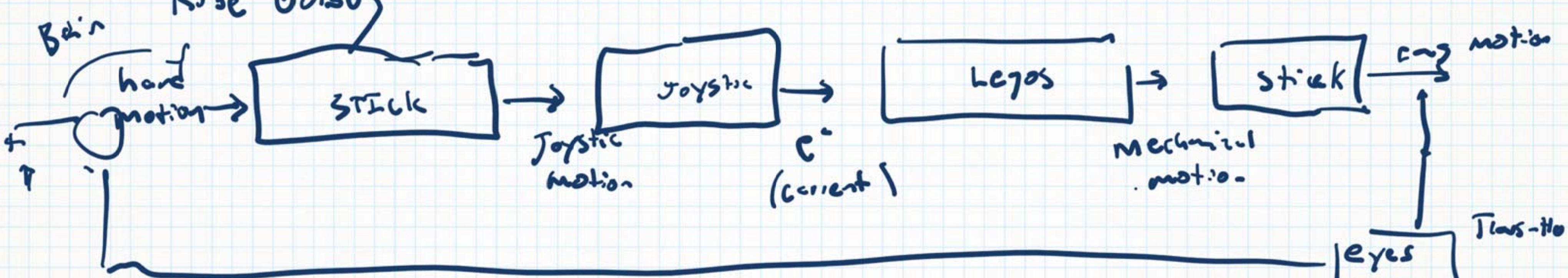
TRANSFER FUNCTIONS (Chp. 4)

↳ system function, Network function

Dr. Reuel's definition : relates the output you want to control, to one of the inputs

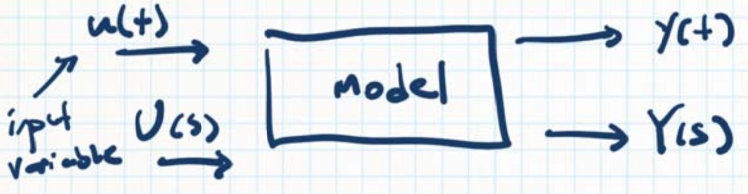


Rube Golberg



Process Control → Transfer function : Laplace domain

(Linear functions)



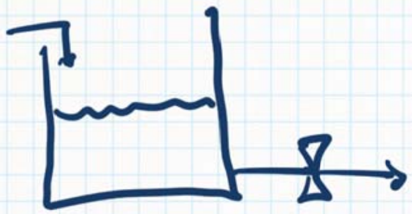
$$G_s = \frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)}$$

→ deviation variable

: you do not need to carry around the S.S. / initial conditions in Laplace space

$$y^* = y' = y_{(t)} - \bar{y} \qquad u^* = u' = u_{(t)} - \bar{u}$$

Example 1 : 1 input variable



$$A\frac{dh}{dt} = q_i - \frac{1}{R_v} h \quad \Leftarrow \text{ODE model}$$

① Deviation variables

$$h^* = h - \bar{h}$$

$$q^* = q - \bar{q}$$

S.S. $$0 = \bar{q}_i - \frac{1}{R_v}\bar{h}$$

$$A\frac{dh}{dt} = (q_i - \bar{q}_i) - \frac{1}{R_v}(h - \bar{h})$$

constant

$$\frac{dh^*}{dt} = \frac{d(h-\bar{h})}{dt} = \frac{dh}{dt}$$

$$\rightarrow A\frac{dh^*}{dt} = q^* - \frac{1}{R_v}h^*$$

② Laplace of both sides

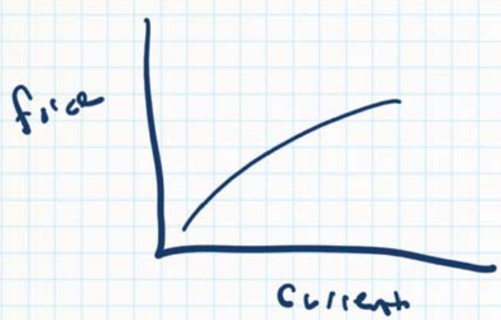
$$A\left[sH^*(s) - \cancel{h^*(0)}^{\,0}\right] = Q^*(s) - \frac{1}{R_v}H^*(s)$$

$$G_{(s)} = \frac{\text{output}}{\text{input}} = \frac{H^{*}_{(s)}}{Q^{*}_{(s)}} = \frac{R_v}{AR_v s + 1}$$

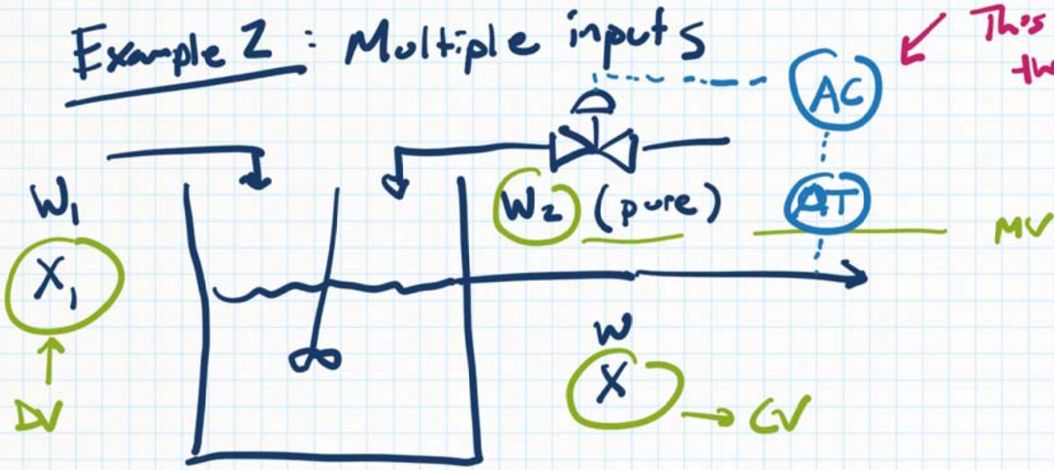
Significance?

$$Q_{i(s)} = \frac{M}{s} \leftarrow \text{pulse of magnitude } M$$

$$H^{*}_{(s)} = \left(\frac{R_v}{AR_v s + 1}\right)\left(\frac{M}{s}\right)$$



Example 2 : Multiple inputs



This is where the transfer function is used

V is constant
ρ is constant
W_1 : mass flow in is constant
initially @ S.S.
uniform mixing

Mass balance

$$\rho \frac{dV}{dt} = W_1 + W_2 - W = 0$$

$$W = W_1 + W_2$$

component balance

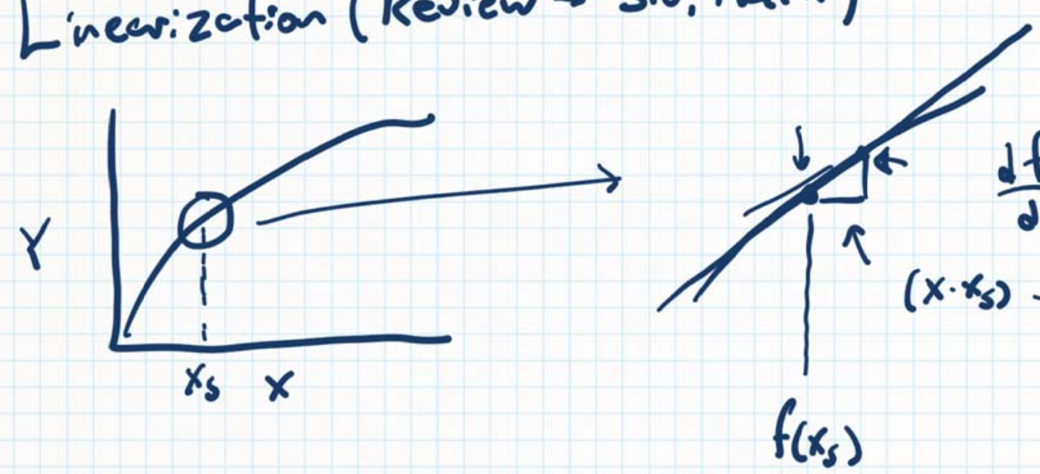
$$\rho V \frac{dx}{dt} = W_1 x_1 + W_2 - W x$$

$$= W_1 x_1 + W_2 - W_1 x - \boxed{W_2 x}$$

$W_2(t)$ $x(t)$

Non-Linear

Linearization (Review → 310, Math)



$\frac{df}{dx} = \frac{\text{rise}}{\text{run}}$

$(x \cdot x_s)$

Taylor series expansion

x^* higher order

one variable $\boxed{f_{(x)} = f(x_s) + \frac{df}{dx}(x - x_s)} + \mathcal{O}$

$f : \sqrt{h}$

two variables $f_{(x,y)} = f(x_s, y_s) + \left(\frac{\delta f}{\delta x}\right)\Big|_{x_s, y_s} (x^*) + \left(\frac{\delta f}{\delta y}\right)\Big|_{x_s y_s} y^*$

$$f_{(x,y)} = f_{(W_2, x)} = W_2 x$$

$\bar{\ } = ss$

2 Linearized

$$W_2 x \cong \bar{W}_2 \bar{x} + \bar{W}_2 x^* + \bar{x} y^*$$

3 Write ODE in terms of deviation variables

$$\rho V \frac{dx^*}{dt} = W_1 x_1^* + W_2^* - W_1 x^* - \bar{W}_2 x^* - \bar{x} W_2^*$$

4 Standard form

$$\rho V \frac{dx^*}{dt} + (W_1 + \bar{W}_2) \underset{\text{output}}{x^*} = W_1 x_1^* + (1 - \bar{x}) W_2^*$$

$$\underbrace{\frac{\rho V}{(W_1 + \bar{W}_2)}}_{\tau_p} \frac{dx^*}{dt} + \underset{\uparrow\ \text{CV}}{x^*} = \underbrace{\frac{W_1}{(W_1 + \bar{W}_2)}}_{K_D \text{ (gain)}} \underset{\uparrow\ \text{DV}}{x_1^*} + \underbrace{\frac{1 - \bar{x}}{(W_1 + \bar{W}_2)}}_{K_p} \underset{\uparrow\ \text{MV}}{W_2^*}$$

$\mathcal{L}(\text{ODE}) \Rightarrow \quad s\tau_p X^*(s) + X^*(s) = K_D X_1^*(s) + K_p W_2^*(s)$

$$X^*(s) = \underbrace{\frac{K_D X_1^*(s)}{s\tau_p + 1}}_{\text{Transfer function for } X_1(s) \text{ DV}} + \underbrace{\frac{K_p W_2^*(s)}{s\tau_p + 1}}_{\text{Transfer function for } W_2 \text{ (MV)}}$$

$$\boxed{\frac{k_p}{\tau_p s + 1}}$$

gain

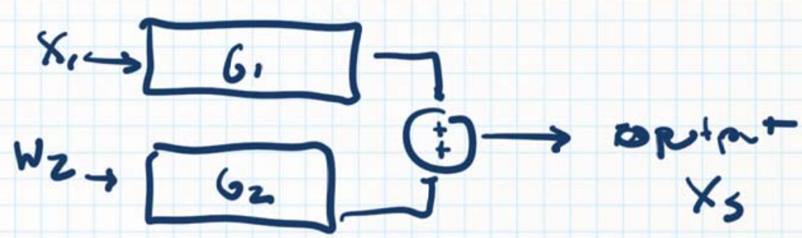
Time constant

$$G_1(s) = \left.\frac{X^*(s)}{X_1^*(s)}\right|_{W_2^* = 0} = \frac{K_D}{(s\tau_p + 1)}$$

$$G_2(s) = \left.\frac{X^*(s)}{W_2^*(s)}\right|_{X_1^* = 0} = \frac{K_p}{(s\tau_p + 1)}$$

$$X^*(s) = G_1(s) X_1^*(s) + G_2(s) W_2^*(s)$$

independent variables



Lecture 6: Transfer Functions + First Order Processes

(forcing functions)

$U_{(s)} \longrightarrow \boxed{\text{model}} \longrightarrow Y_{(s)}$

$$\frac{Y_{(s)}}{U_{(s)}} = G_{(s)}$$

Transfer function

$$\boxed{Y_{(s)} = \frac{K}{\tau s + 1} U_{(s)}}$$

$\lim_{s \to 0} \Rightarrow K$

what links input to output at s.s.

gain

Forcing Function

[1] Step Input

$$u_{step}(t) = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases}$$

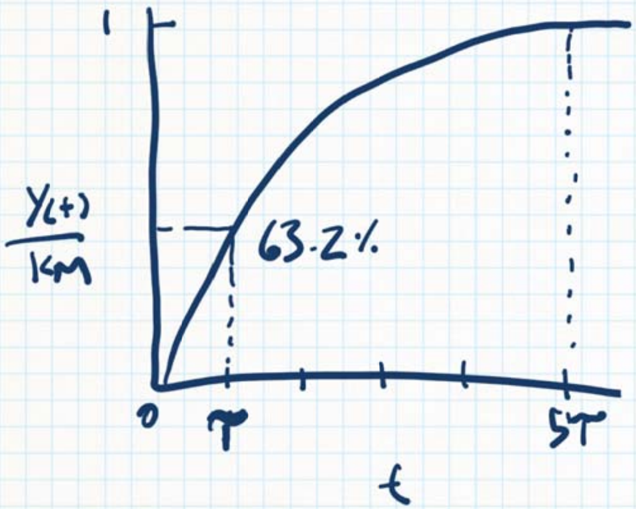
$$\downarrow$$

$$U_{(s)} = \frac{M}{s}$$

$$Y_{(s)} = \frac{K}{(\tau s + 1)} \frac{M}{s}$$

$$Y_{(s)} = KM \left(\frac{1}{(\tau s + 1) s} \right)$$

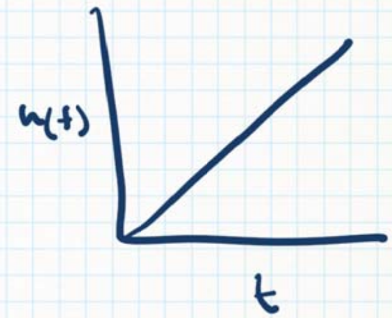
$$y(t) = (1 - e^{-t/\tau}) KM$$



$5\tau \rightarrow 99.3\%$ to s.s.

$\boxed{2}$ Ramp Input

$$u_{ramp}(t) = \begin{cases} 0 & t < 0 \\ at & t \leq 0 \end{cases}$$



$$U_{ramp}(s) = \frac{a}{s^2}$$

$$Y(s) = \frac{Ka}{(\tau s+1)s^2} \Rightarrow y(t)$$

$$Y(s) = \frac{\alpha_1}{(\tau s+1)} + \frac{\alpha_2^{*}}{s} + \frac{\alpha_3^{*}}{s^2}$$

$$\left.\frac{\cancel{(\tau s+1)}Ka}{\cancel{(\tau s+1)}s^2}\right|_{s=\frac{-1}{\tau}} = \alpha_1$$

$$\alpha_1 = Ka\tau^2$$

$$\alpha_3 = \left.\frac{Ka}{(\tau s+1)}\right|_{s=0} = Ka$$

$$\frac{Ka}{(\tau s+1)s^2} = \frac{Ka\tau^2}{(\tau s+1)} + \frac{\alpha_2}{s} + \frac{Ka}{s^2} \longleftarrow s=1, \text{ solve for } \alpha_2$$

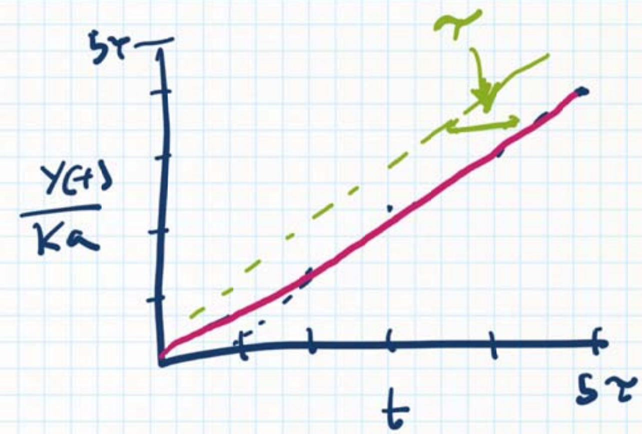
$$\alpha_2 = -Ka\tau$$

$$Y(s) = \underbrace{\frac{Ka\tau^2}{\tau s+1}}_{\#6} - \underbrace{\frac{Ka\tau}{s}}_{\#2} + \underbrace{\frac{Ka}{s^2}}_{\#3}$$

$$\frac{1}{\tau}e^{-t/\tau}(Ka\tau^2) - Ka\tau + Kat$$

$$Y(t) = Ka\tau\left(e^{-t/\tau} - 1\right) + Kat$$

What happens $t \gg \tau$ $\quad Y(t) \cong Kat$

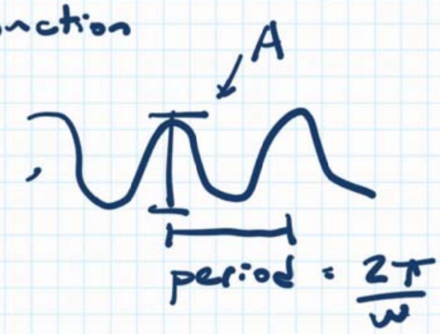


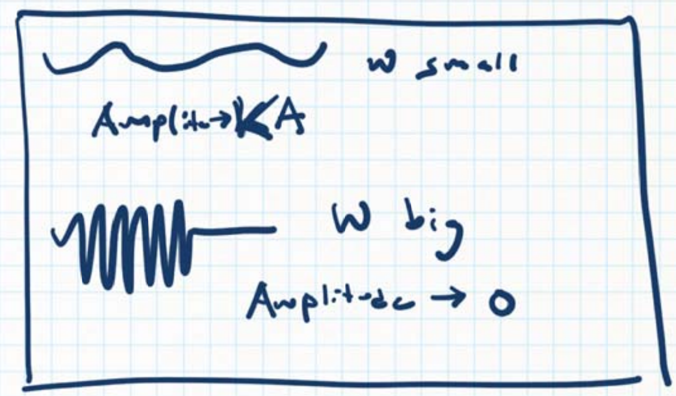
3 (First order Forcing Function)

ω = freq (in radians)

Sinusoidal Forcing Function

$$u_{sin}(t) = A\sin(\omega t)$$





$$U_{sin}(s) = \frac{A\omega}{(s^2+\omega^2)}$$

$$Y(s) = \frac{KA\omega}{(\tau s+1)(s^2+\omega^2)} = \frac{KA}{\omega^2\tau^2+1}\left(\frac{\omega\tau^2}{(\tau s+1)} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right)$$

$$Y(t) = \frac{KA}{\omega^2\tau^2+1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos(\omega t) + \sin(\omega t)\right)$$

$t \to \infty$

$$Y(t) = \frac{KA}{\omega^2\tau^2+1}e^{-t/\tau} + \underbrace{\boxed{\frac{KA}{\sqrt{\omega^2\tau^2+1}}}}_{\text{Amplitude of oscillation at s.s.}}\sin\left(\omega t + \phi\right)$$

$-\tan^{-1}(\omega t)$

Multiplicative Property of Transfer Functions

[dependent variables]

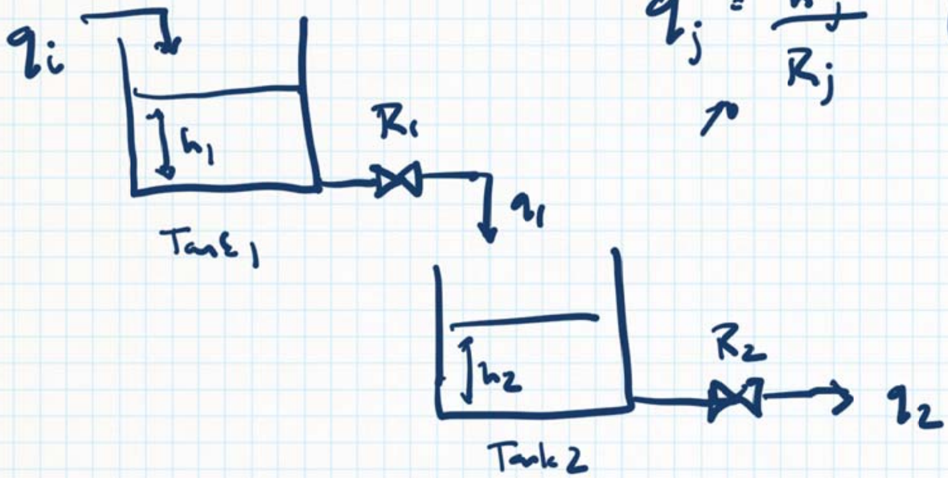
$$U(s) \rightarrow \boxed{G_1(s)} \xrightarrow{Y_1(s)} \boxed{G_2(s)} \xrightarrow{Y_2(s)}$$

$$Y_1(s) = G_1(s)\,U(s)$$

$$Y_2(s) = G_2(s)\,Y_1(s)$$

$$Y_2(s) = G_1(s)\,G_2(s)\,U(s)$$

$$\frac{Y_2(s)}{U(s)} = G_p(s) = G_1(s)\,G_2(s)$$



$q_j = \dfrac{h_j}{R_j}$ (Valve, linear dependence on height)

Tank 1 $\quad A_1 \dfrac{dh_1}{dt} = q_i - q_1 = q_i - \dfrac{h_1}{R_1}$

$$A_1 \frac{dh_1^*}{dt} = q_i^* - \frac{h_1^*}{R_1}$$

$$\mathcal{L}(\text{ODE}) \Rightarrow s A_1 H_1^*(s) + \frac{H_1^*(s)}{R} = Q_i^*(s)$$

$$q_i \rightarrow \underset{\text{ODE}}{\boxed{\text{Tank 1}}} \xrightarrow{h_1} \boxed{\text{Valve1}} \xrightarrow{q_1} \underset{\text{ODE}}{\boxed{\text{Tank2}}} \xrightarrow{h_2} \boxed{\text{Valve2}} \xrightarrow{q_2}$$

$$H^*(s) = \frac{R_1}{(A_1R_1s+1)} Q_i^*(s)$$

K

$$G_{t1} = \frac{R_1}{(\underbrace{A_1R_1}_{\tau}s+1)}$$

Valve 1 $\quad Q_1^* = \dfrac{H_1^*}{R_1} \qquad G_{v_1} = \dfrac{1}{R_1}$

$$G_{t2} = \frac{R_2}{(A_2R_2s+1)} \qquad G_{v_2} = \frac{1}{R_2}$$

$$G_p = \frac{Q_2^*}{Q_i^*} = \underbrace{\left(\frac{Q_2^*}{\cancel{H_2^*}}\right)}_{\text{Valve 2}}\underbrace{\left(\frac{\cancel{H_2^*}}{\cancel{Q_1^*}}\right)}_{\text{Tank 2}}\underbrace{\left(\frac{\cancel{Q_1^*}}{\cancel{H_1^*}}\right)}_{\text{Valve 1}}\underbrace{\left(\frac{\cancel{H_1^*}}{Q_i^*}\right)}_{\text{Tank 1}}$$

$$G_p = G_{t_1} G_{v_1} G_{t_2} G_{v_2} = \frac{1}{(A_1 R_1 s + 1)(A_2 R_2 s + 1)}$$

2nd order processes = sometimes oscillate

$$\mathcal{L}(ODE) \Rightarrow \frac{c_1 s + c_0}{(s^2 - d_1 s + d_0)}$$

$\frac{d_1^2}{4} < d_0$, we know that the denominator has complex roots

Appendix L (Book) → Completing the Square

$$= \frac{a_1 (s+b) + a_2}{(s+b^2) + w^2}$$

$$y(t) = a_1 e^{-bt} \cos(\omega t) + \frac{a_2}{\omega} e^{-bt} \sin(\omega t)$$

oscillations

Lecture 7: Second Order Transfer Functions + Stability Analysis (chp. 5 + 6)

$$U(s) \rightarrow \boxed{\overset{G_1}{\frac{K_1}{\tau_1 s+1}}} \xrightarrow{X(s)} \boxed{\overset{G_2}{\frac{K_2}{\tau_2 s+1}}} \xrightarrow{Y(s)}$$

standard form

Transfer function

$$G_{p(s)} = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)} \Rightarrow \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

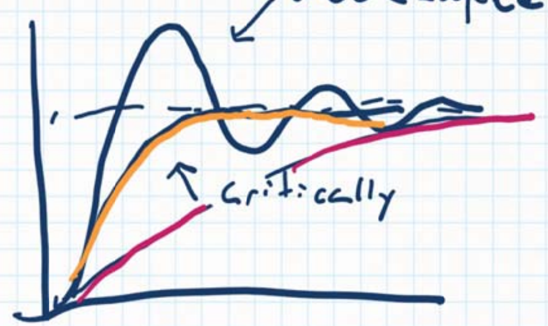
"damping term"

$$G(s) = \frac{Y(s)}{U(s)}$$

$\zeta > 1$ overdamped

$\zeta = 1$ critically damped

$0 < \zeta < 1$ underdamped



$$\tau = \sqrt{\tau_1 \tau_2}$$

$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}}$$

$\boxed{1}$ Step Function ($u_{(s)}$)

$$U_{(s)} = \frac{M}{s}$$

$$Y_{(s)} = \frac{KM}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$$

Case 1 : overdamped ($\zeta > 1$)

$$Y_{(t)} = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right)$$

$$\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}}$$

$$\tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}}$$

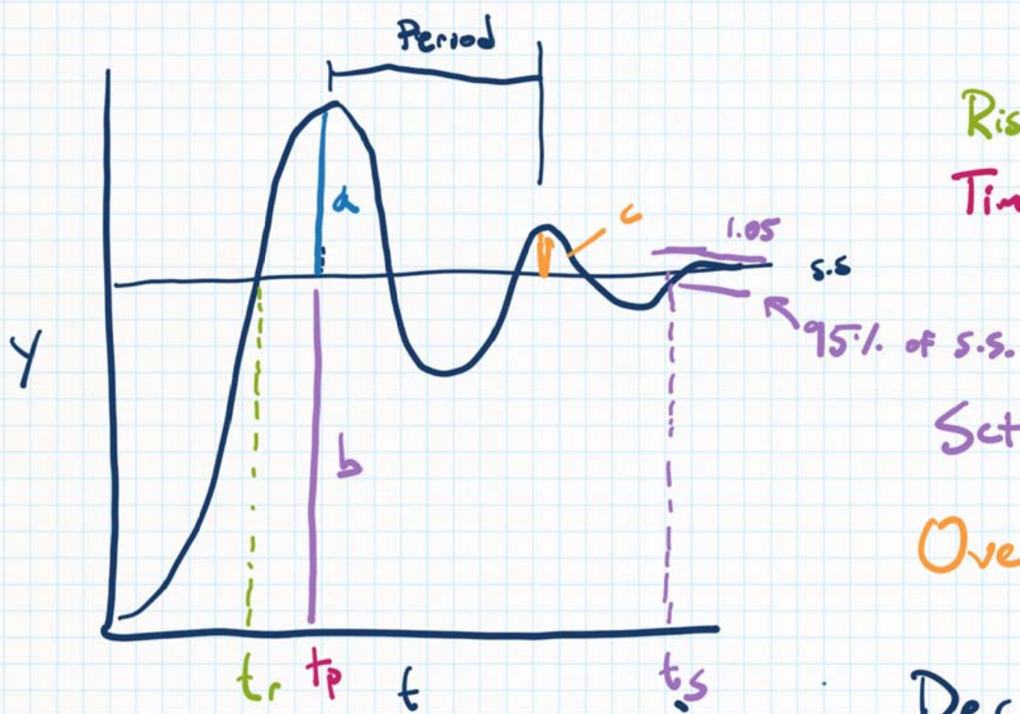
case 2: critically damped $\zeta = 1$

$$Y(t) = KM\left[1 - (1 + t/\tau)e^{-t/\tau}\right]$$

Case 3: underdamped (oscillates.

$0 < \zeta < 1$

$$Y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\}$$



Rise Time (t_r) = first time it reaches S.S.

Time to First Peak (t_p) = first max value

Settling Time (t_s) = time when response is ± 5% of steady state

Overshoot (OS) = $\frac{a}{b}$

Decay Ratio = $\frac{c}{a}$

Period of Oscillation = time between peaks

For 2nd Order Systems (underdamped)

$t_p = \beta \tau$

$$\beta = \frac{\pi}{\sqrt{1-\zeta^2}}$$

$OS = \exp(-\beta \zeta)$

$$K = \frac{\Delta \text{Output}}{\Delta \text{input}}$$

$DR = (OS)^2 = \exp(-2\beta\zeta)$

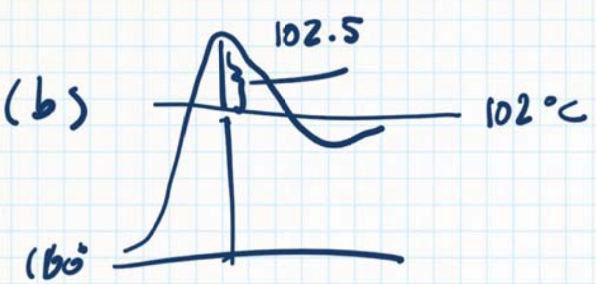
Period $= P = 2\tau/\beta$

Class 7 example Problem

(a) $20 \ \frac{°C}{kg/s}$

to solve for τ

(b)



$a = 0.5$

$b = 2$

$OS = \frac{a}{b} = .25$

$$P = \frac{2\pi \tau}{\sqrt{1-\zeta^2}}$$

$$\zeta = \sqrt{\frac{1}{\frac{\pi^2}{\ln(OS)^2} + 1}}$$

$\zeta = 0.404$

$\tau = 300_s$

$P = 3060_s - 1000_s = 2060_s$

$\boxed{2}$ Sinusoidal forcing function $\quad U(t) = A\sin(\omega t)$

$$Y(t) = \underbrace{\frac{KA}{\sqrt{[1-(\omega\tau)^2 + (2\zeta\omega\tau)^2}}}_{\text{Amplitude} = \hat{A}} \sin(\omega t + \phi)$$

$$\phi = -\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$$

Normalized Amplitude Ratio

$$\frac{\hat{A}}{KA} \leftarrow \text{where does maximum occur}$$

$$\omega_{max} = \sqrt{\frac{1-2\zeta^2}{\tau}} \quad \text{for } 0 < \zeta < 0.707$$

What is the MAX amplitude ratio

$$\left.\frac{\hat{A}}{KA}\right|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Stability Analysis (complex transfer functions)

$$\frac{K}{s(\tau_1 s + 1)(\tau_2^2 s^2 + 2\zeta \tau_2 s + 1)}$$

$s \rightarrow$ constant $\qquad (\tau_1 s+1) \rightarrow e^{-t/\tau_1} \qquad \underbrace{(\tau_2^2 s^2 + 2\zeta \tau_2 s + 1)}_{\text{sin and cos}}$ $\qquad$ [time domain]

→ "response modes" "natural modes"

denominator = roots of the characteristic polynomial

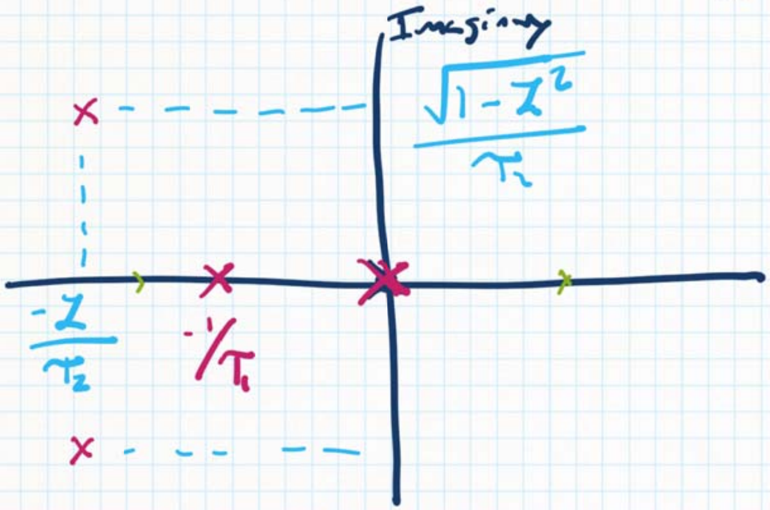
$s_1 = 0$ $\qquad s_3$ and $s_4 \Rightarrow \dfrac{-\zeta}{\tau_2} \pm j \dfrac{\sqrt{1-\zeta^2}}{\tau_2}$

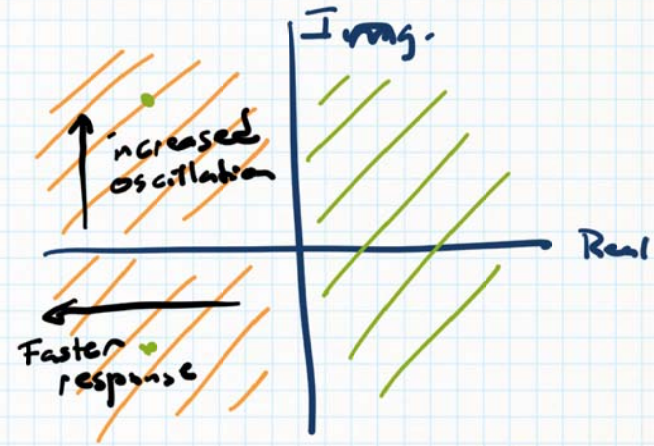
$s_2 = -1/\tau_1$

↖ roots

"poles"

- "complex S plane"
- "pole plot"





$$G(s) = \frac{1}{(s-2)(s+2.5)}$$

Poles? $2 \quad -2.5$

Behavior? → unstable

$$G(s) = \frac{1}{(s^2+4s+5)}$$

$K = 0.2$

$\tau = \sqrt{.2}$

$\zeta = \sqrt{.8} = 0.894$

Quadratic Eqn. $\Rightarrow \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Poles = $-2 \pm j$

Case 1 = unstable. any real (+) pole will produce unstable response

$\zeta < 0 \Rightarrow e^{at}$

$$G(s) = \frac{k(\tau_a s + 1)}{(\tau_1 s + 1)}$$ ← Num dynamic Thursday

Case 2 overdamped $\zeta > 1$

- Poles on Real negative axis, overdamped, non-oscillatory
- slower as you get closer to origin

$$P_1, P_2 = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$$

Case 3 critically damped

$\zeta = 1$ $P_1 = P_2 = -1/\tau$ no oscillation

Case 4 underdamped

$0 < \zeta < 1$

$$P_1, P_2 = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$$ complex conjugates

$\zeta \to 1$, less oscillation

Case 5 undamped

$\zeta = 0$

poles only have imaginary component

→ oscillate FOREVER

$$\frac{Y(s)}{U(s)} = G(s)$$

$$\frac{M}{s}$$

$$G(s) = \frac{K}{(\tau^2 s^2 + 2\zeta s + 1)} \frac{M}{s}$$

$$Y(t) =$$

Lecture 8: Numerator Dynamics, Time Delays, Approximations

(chp. 6)

$$\tau_1 \frac{dy}{dt} + y = K\,u(t)$$

$$= K\left(\tau_a \frac{du}{dt} + u\right)$$

"lead-lag" forcing function

" derivative filter "

$$G(s) = \frac{K(\tau_a s + 1)}{\tau_1 s + 1}$$

s term in Numerator

" Numerator Dynamics "

roots ⇒ Zeros

poles

standard form

$$G(s) = \frac{K(\tau_a s + 1)(\tau_b s + 2)\ldots}{(\tau_1 s + 1)(\tau_2 s + 1)\ldots}$$

$$G(s) = \frac{b_m (s - z_1)(s - z_2)\ldots(s - z_m)}{a_n (s - p_1)(s - p_2)\ldots(s - p_n)}$$

$$z_1 = -1/\tau_a \qquad p_1 = -1/\tau_1$$

General Properties of "Zeros"

- → Presence or absence of system zeros has <ins>NO</ins> effect on the # and location of poles <ins>unless</ins> they directly cancel

- → The zeros affect the coefficients of the response modes "The first transient behavior"

- → # of zeros $\leq$ # of poles

- → Real processes have "higher order dynamics" causes your process to have some degree inertia

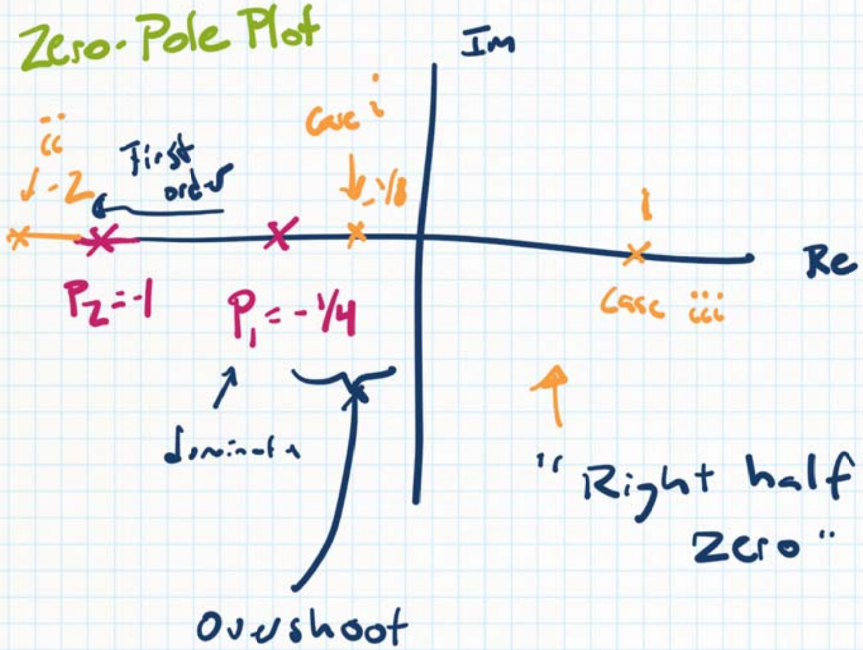
$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$\tau_1 = 4$

$\tau_2 = 1$

varies τ_a

Zero-Pole Plot



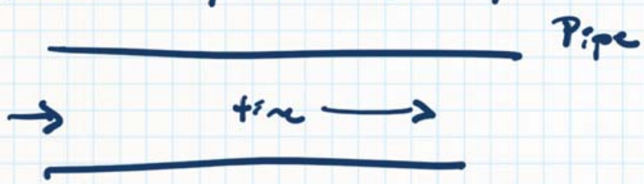
Case i $\Rightarrow \tau_a = 8$

Case ii $\Rightarrow \tau_a = 0.5$

Case iii $\Rightarrow \tau_a = -1$

"Right half plane zero" = inverse dynamics

Process w/ Time Delays



" distance velocity lag "
" transportation lag "
" transport delay "
" dead time "

$$\theta = \frac{\text{length of pipe}}{\text{fluid velocity}}$$

$$\theta = \frac{\text{volume of pipe}}{\text{volumetric flow rate}}$$

$$\mathcal{L} \Rightarrow e^{-\theta s}$$

$$G(s) = \frac{Ke^{-\theta s}}{\tau s+1}$$

First order plus dead time (FOPDT)

irrational poles/zeros

$$G(s) = \frac{Ke^{-\theta s}}{(\tau_1 s+1)(\tau_2 s+1)}$$

(SOPDT)

TAYLOR Series of

$$e^{-\theta s} = \underbrace{1 - \theta s} + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} - \frac{\theta^5 s^5}{5!} + \cdots$$

1/1 Padé approximation of

$$e^{-\theta s} = G_1(s) = \frac{1 - \frac{\theta}{2} s}{1 + \frac{\theta}{2} s} = \underbrace{1 - \theta s} + \frac{\theta^2 s^2}{2} - \frac{\theta^3 s^3}{4}$$

Taylor approx for Numerator

$$e^{-\theta s} \cong (1 - \theta s) \quad \text{and} \quad e^{\theta s} = (1 + \theta s)$$

Taylor approx for denominator

$$\frac{1}{(1 + \theta s)} = \frac{1}{e^{\theta s}} = e^{-\theta s}$$

$\boxed{2}$ Skogestad's 1/2 Rule

(1) Keep dominant τ_1 (largest) for FOPDT or dominant τ_1, τ_2 for SOPDT (denominator)

(2) Add 1/2 of the largest, neglected τ to smaller of the retained τ in denominator

(3) Use the remaining 1/2 τ in an approximation for time delay

(4) Approx. any remaining terms w/ Taylor's approx.

$\boxed{\text{Ex1}}$

$$G(s) = \frac{K(-0.1s+1)}{(5s+1)(3s+1)(0.5s+1)} \Rightarrow \text{FOPDT}$$

$\tau_1 = 5$ $\quad \tau_2 = 3 \quad \tau_3 = 0.5$

dominant τ

(a) Taylor series approx

Den. $\quad \dfrac{1}{(3s+1)} \cong e^{-3s} \qquad \dfrac{1}{(0.5s+1)} \cong e^{-0.5s}$

Num. $\quad (-0.1s+1) \cong e^{-0.1s}$

$$G(s) \cong \frac{Ke^{-0.1s}}{(5s+1)} e^{-3s} e^{-0.5s} = \frac{Ke^{-3.6s}}{5s+1}$$

FOPDT approx. using Taylor series

(b) Skogestad 1/2 method

$\tau_1 = 5$ dominant

$\tau_2 = 3 \leftarrow \quad \frac{1}{2} \cdot \tau_2 = 1.5$

$\frac{1}{2}\tau_2$

$$G(s) \approx \frac{Ke^{-0.1s}}{(6.5s+1)(0.5s+1)} e^{-1.5s}$$

$\frac{1}{2}\tau_2$

Skogestad's approx

$$\frac{1}{0.5s+1} \approx e^{-0.5s}$$

$$G(s) \approx \frac{Ke^{-2.1s}}{6.5s+1}$$

$$G(s) = \frac{K(1-s)e^{-s}}{(12s+1)(3s+1)(0.2s+1)(0.05s+1)}$$

$\hookrightarrow$ FOPDT

1.5

$\tau = 13.5$

e^{-s}

$$\theta = 1 + \frac{3}{2} + \underbrace{0.2 + 0.05}_{\text{Denominator}} + 1$$

current θ

1/2 rule

$= 3.75$

$$G(s) = \frac{Ke^{-3.75s}}{(13.5s+1)}$$

SOPDT $\rightarrow$ $$G(s) = \frac{Ke^{-\theta s}}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$\tau_1 \leftarrow 12 \qquad \tau_2 \leftarrow 3+0.1$$

$$\tau_3 = 0.2$$

$$\tfrac{1}{2}\tau_3 = 0.1$$

$$G(s) \cong \frac{K(1-s)e^{-s}}{(12s+1)(3.1s+1)} e^{-0.1s} e^{-0.05s} \cong \frac{K(1-s)e^{-1.15s}}{(12s+1)(3.1s+1)}$$

$$G(s) \cong \frac{Ke^{-2.15s}}{(12s+1)(3.1s+1)}$$

Lecture 9 Chapter 7

$$U(s) \longrightarrow \boxed{\frac{K}{\tau s+1}} \longrightarrow Y(s)$$

↑
process modeling

$$U(s) \longrightarrow \boxed{\text{Black Box}} \longrightarrow Y(s)$$

process identification
(System)

↳ empirical (real world data)

Simpler

Solved in real time

→ sometimes we know a general form of the model

⚠ DO NOT EXTRAPOLATE

General Steps

→ Select input + output variable (SISO)

→ determine model form

→ fit unknown parameters [310 → Matlab, Excel]

→ Test your model

  → old historic data

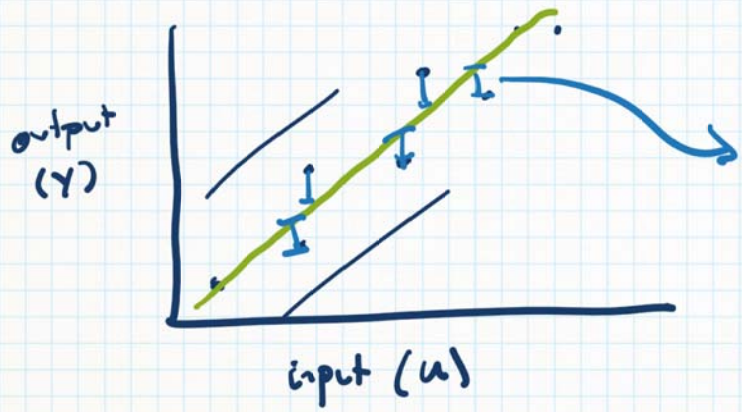
  - New data ("validation data")

→ Check statistical criteria

$\boxed{\text{Method 1}}$ Linear Regression

$Y = \beta_1 + \beta_2 u$

$\beta_1 \rightarrow$ 95% confidence



$Y_{model} - Y_{data} = \varepsilon$

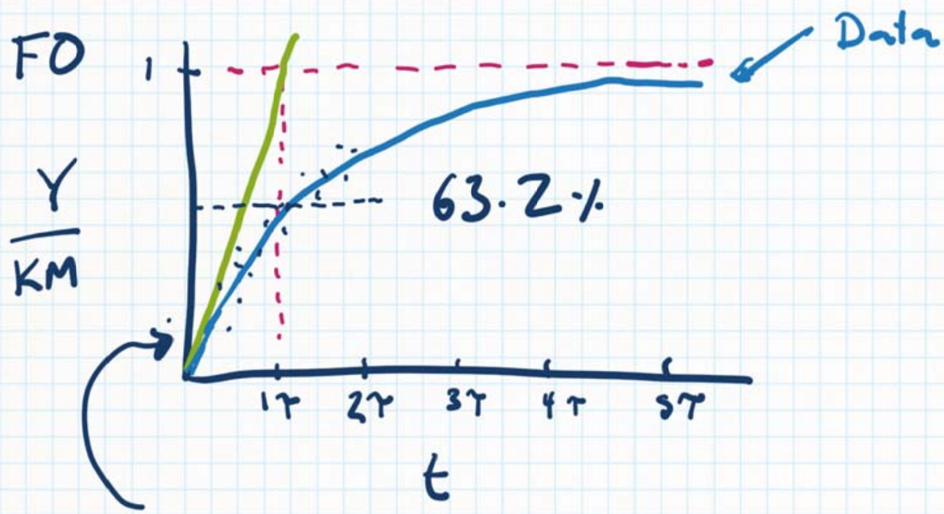
$SSE = \sum_{i=1}^{N} \varepsilon_i^2$

Low

<u>Method 2</u>: Graphical Correlations

"If you want to impress your boss during a meeting"

"Party Tricks"



Trick 1 : initial slope = $\frac{1}{\tau}$

$$\tau \frac{dy}{dt} + y = k\,U(s)$$

Reasons this is NOT real

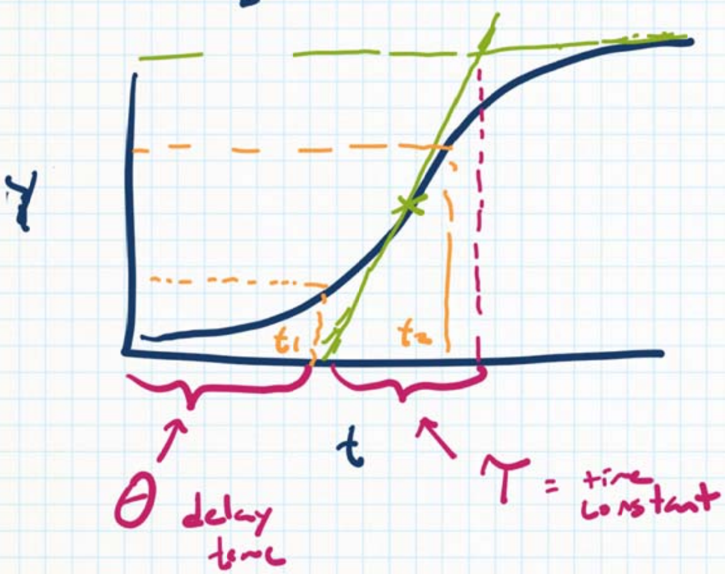
→ System inertia

→ Noise

- unknown disturbances

→ No perfect step input

FOPDT [Graphical]



$$G(s) = \frac{K e^{-\theta s}}{(\tau s + 1)}$$

$$K = \frac{\Delta \text{Output}}{\Delta \text{input}}$$

← Problem is one point fit

Two point Fit

find t_1 => 35.3 % response

find t_2 => 85.3 % response

$$\theta = 1.3 t_1 - 0.29 t_2$$

$$\tau = 0.67 (t_2 - t_1)$$

SO

$$G(s) = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

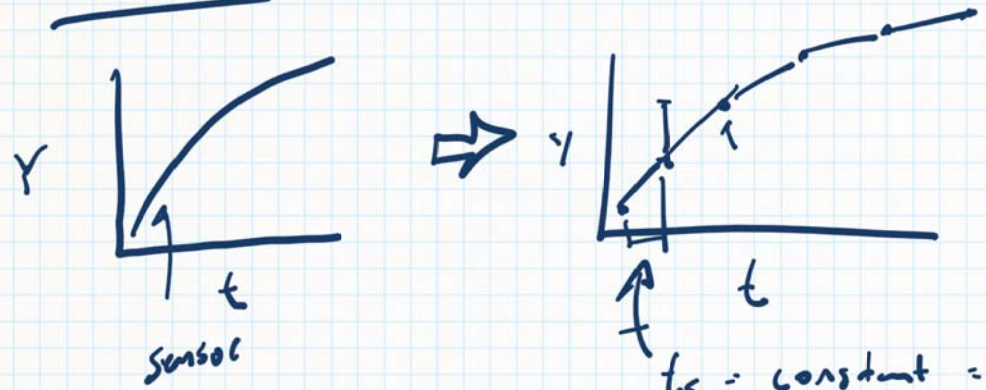
zeta

Largest time constant = dominant

SOPDT

$$G(s) = \frac{Ke^{-\theta s}}{(\tau^2 s^2 + 2\zeta\tau s + 1)}$$

Topic #3 Discrete Time Models



t_s = constant = sampling time

t	y

finite difference method

$\frac{dy}{dt} \leftarrow$ slope

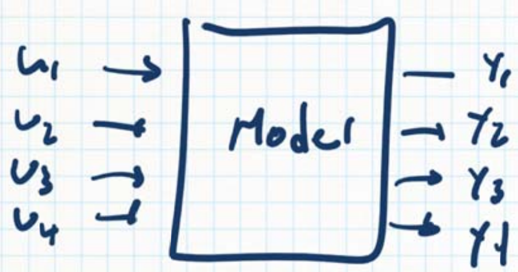
$$\frac{dy}{dt} \simeq \frac{Y(k) - Y(k-1)}{\Delta t}$$

① Soft Sensors

- physical parameters inside system
- component amount

Spetra

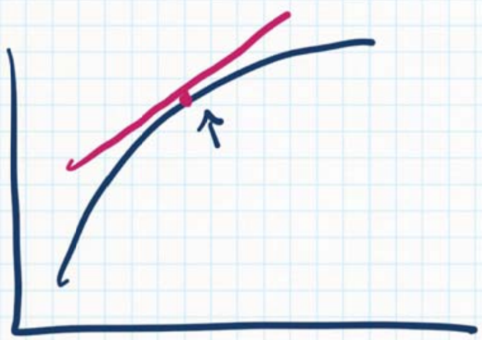
↳ a lot of data



② Neural Networks

Exam Review

Topic 1 → Linearization



consider a flow system

Flow = 1
height = 1 @ s.s.

change Flow = 2
height = 4

$$\frac{(4-1)}{(2-1)} = K = 3$$

change Flow = 3
height = 9

$$\frac{(9-1)}{(3-1)} = K = 4$$

Example ODE w/ Non-linear term

$$\frac{dx}{dt} = -x^2 + \sqrt{u}$$

↳ taylor series (truncated)

deviation variable

$$\frac{dy}{dt} = f(\cancel{x}, u) \approx f(\bar{y}, \bar{u}) + \frac{\delta f}{\delta y}\bigg|_{\bar{y},\bar{u}} (y - \bar{y}) + \frac{\delta f}{\delta u}\bigg|_{\bar{y},\bar{u}} (u - \bar{u})$$

$$\frac{dy^*}{dt} = \alpha y' + \beta u' + \dots \gamma d'$$

$$\alpha = \frac{\delta f}{\delta y}\bigg|_{\bar{y},\bar{u},\bar{d}} \qquad \beta = \frac{\delta f}{\delta u}\bigg|_{\bar{y},\bar{u},\bar{d}} \qquad \gamma = \frac{\delta f}{\delta d}\bigg|_{\bar{y},\bar{u},\bar{d}}$$

Example 1 : Linearization

$$\frac{dx}{dt} = -x^2 + \sqrt{u}$$

$$\alpha = \frac{\delta f}{\delta x} = \frac{\delta(-x^2 + \sqrt{u})}{\delta x} = -2x$$

$$\beta = \frac{\delta f}{\delta u} = \frac{\delta(-x^2 + \sqrt{u})}{\delta u} = \frac{1}{2}\frac{1}{\sqrt{u}}$$

$$\frac{dx}{dt} \cong f(\bar{x}, \bar{u}) + \alpha(x - \bar{x}) + \beta(u - \bar{u})$$

$f(\bar{x}, \bar{u}) = 0$ (problem statement) $\rightarrow$ $x = 0$, $u = 0$

$\bar{u} = 16$ what is $\bar{x}$?

$$\frac{dx}{dt} = 0 = -\bar{x}^2 + \sqrt{\bar{u}} \qquad \bar{x} = 2$$

$$\frac{dx}{dt} = -4(x - 2) + \frac{1}{8}(u - 16)$$

Example 2: Linearization

$$m\frac{d(v(t))}{dt} = \left[F_p u(t) - \frac{1}{2}\rho A C_d v(t)^2\right]$$

$$\frac{dv(t)}{dt} \simeq \alpha\left(u(t) - 40\right) + \beta\left(v(t) - \bar{v}\right)$$

$$\alpha \leftarrow \frac{\delta}{\delta u} \qquad \beta \leftarrow \frac{\delta}{\delta v}$$

what is $\bar{v}$?

$$\alpha = \frac{F_p}{m} \qquad \beta = -\frac{\rho A C_d}{m}\bar{v}$$

$$0 = F_p \bar{u} - \frac{1}{2}\rho A C_d \bar{v}^2$$

$$\bar{v} = \sqrt{\frac{2 \cdot F_p \bar{u}}{\rho A C_d}} = \sqrt{\frac{2 \cdot 30\,N/\% \cdot 40\%}{1.225\,\frac{kg}{m^3} \cdot 5m^2 \cdot 0.24}} = 40.4\,m/s$$

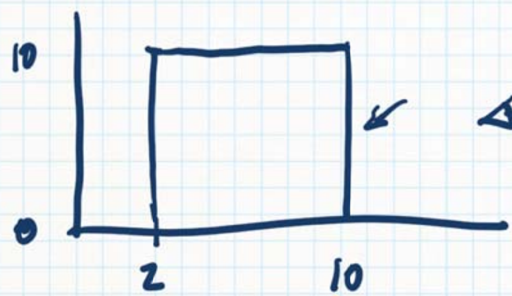
#12 Forcing Functions

$$\underset{U(s)}{u(t)} \longrightarrow \boxed{\text{model}} \longrightarrow \underset{Y(s)}{y(t)}$$

$$G(s) = \frac{Y(s)}{U(s)} \Rightarrow Y(s) = \underbrace{G(s)U(s)}_{\text{inverse Laplace}}$$

Forcing Function

$y(t)$



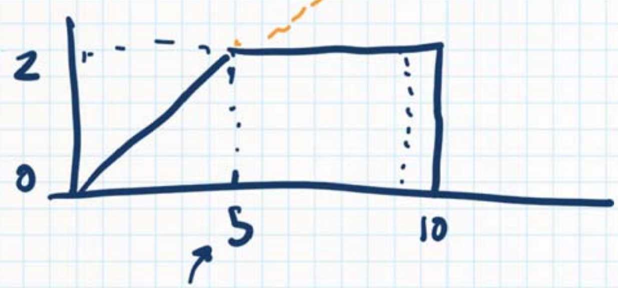
$u(t)$

"switch"

$$u(t) = 10S(t-2) - 10S(t-10)$$

"heaviside"

$$U(s) = \frac{10}{s}e^{-2s} - \frac{10}{s}e^{-10s}$$



$$u(t) = \frac{2}{5}t - \frac{2}{5}(t-5)\,S(t-5) - 2S(t-10)$$

$$U(s) = \frac{2}{5s^2} - \frac{2}{5s^2}e^{-5s} - \frac{2}{s}e^{-10s}$$

$$G(s) = \frac{K}{\tau s+1}$$